Differential Equations (Separable Variables and Homogeneous)

Solve the following differential equation:

1. $(3 + x^2)dy - 2xydx = 0$
2. $3e^x \tan(y) + (1 - e^x)(\sin(y))^2 \dfrac{dy}{dx} = 0$
3. $(x + y)dx + xdy = 0$
4. $\tan(x)(\sin(y))^2 dx + (\cos(x))^2 \cot(y)dy = 0$

Solution

Note: When solving the different differential equations, I will always assume that the arguments of the natural logarithms are strictly positive, so $\int \frac{1}{x} dx = \ln(|x|) + C$, to avoid complex notation $\ln(x) = \ln(|x|)$, moreover, whenever I divide by a term, I will also assume that this term is different from 0. Finally, I will only add the constant of integration in the final step of the solution.

1. This is a differential equation with separable variables:

$$(3 + x^2)dy = 2xydx$$

$$\frac{1}{y}dy = \frac{2x}{3 + x^2}dx$$

I integrate both sides with respect to x:

$$\ln(y) = \int \frac{2x}{3 + x^2}dx$$

Using substitution:

$$u = 3 + x^2$$

$$du = 2xdx$$

$$\ln(y) = \int \frac{du}{u}$$

$$\ln(y) = \ln(u) + C$$

$$\ln(y) = \ln(3 + x^2) + C$$

$$y = (3 + x^2)e^C$$

$$y = (3 + x^2)K$$

2. This is a differential equation with separable variables:

$$3e^x \tan(y) + (1 - e^x)(\sin(y))^2 \frac{dy}{dx} = 0$$

$$3e^x \tan(y) = -(1 - e^x)(\sin(y))^2 \frac{dy}{dx}$$

$$\frac{-3e^x}{1 - e^x}dx = \frac{(\sin(y))^2}{\tan(y)}dy$$

Remember the trigonometric identities: $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

$$\frac{-3e^x}{1 - e^x}dx = \sin(y)\cos(y)dy$$

We integrate both sides with respect to x. First on the left side:

$$\int \frac{-3e^x}{1 - e^x}dx$$

Let's make a substitution: $u = 1 - e^x$, so $du = -e^x$, then:

$$3\int \frac{du}{u} = 3\ln(u) = 3\ln(1 - e^x)$$

On the right side:

$$\int \sin(y)\cos(y)dy$$

Let's make a substitution: $u = \sin(y)$, so $du = \cos(y)dx$.

$$\int udu = u^2 + C = +C$$

Going back to the equation:

$$3\ln(1-e^x) = \frac{(\sin(y))^2}{2} + C$$

3. This is a differential equation with separable variables:

$$\frac{\tan(x)}{(\cos(x))^2}dx = -\frac{\cot(y)}{(\sin(y))^2}dy$$

Remember the trigonometric identity: $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and $\cot(y) = \dfrac{\cos(y)}{\sin(y)}$

$$\frac{1}{(\cos(x))^2}\frac{\sin(x)}{\cos(x)}dx = -\frac{1}{(\sin(y))^2}\frac{\cos(y)}{\sin(y)}dy$$

$$\frac{\sin(x)}{(\cos(x))^3}dx = -\frac{\cos(y)}{(\sin(y))^3}dy$$

I integrate both sides with respect to x, on the right side:

$$\int -\frac{\cos(y)}{(\sin(y))^3}dy$$

I substitute, $u = (\sin(y))$ so: $du = \cos(y)dy$

$$-\int \frac{du}{u^3} = \frac{u^{-2}}{2} =$$

I integrate on the left side using the substitution: $u = \cos(x)$, so $du = -\sin(x)dx$

$$\int \frac{\sin(x)}{(\cos(x))^3}dx = \int \frac{-du}{u^3} = -\frac{u^{-2}}{-2} = \frac{(\cos(x))^{-2}}{2}$$

Finally:

$$\frac{(\cos(x))^{-2}}{2} = \frac{(\sin(y))^{-2}}{2} + C$$

4. This is a homogeneous differential equation. I divide everything by x^1:

$$\frac{x+y}{x}dx + dy = 0$$

$$(1 + y/x)dx + dy = 0$$

I make the substitution: $v = y/x$. So $y = vx$, and by taking the total derivative: $dy = vdx + xdv$

$$(1+v)dx + (vdx + xdv) = 0$$

$$dx + vdx + vdx + xdv = 0$$

$$(1+2v)dx = -xdv$$

$$\frac{1}{x}dx = \frac{-1}{1+2v}dv$$

I integrate both sides:

$$\ln(x) = \int \frac{-1}{1+2v}dv$$

I use substitution: $h = 1 + 2v$, so: $dh = 2dv$ and $dh/2 = dv$.

$$\ln(x) = \int \frac{-1}{h}\frac{dh}{2}$$

$$\ln(x) = \frac{-1}{2}\ln(h) + C$$

$$\ln(x) = \frac{-1}{2}\ln(1+2v) + C$$

$$\frac{-(\ln(x) - C)}{2} = \ln(1+2v)$$

$$\frac{(e^{\frac{-(\ln(x)-C)}{2}} - 1)}{2} = v$$

Substituting back for v:

$$\frac{(e^{\frac{-(\ln(x)-C)}{2}} - 1)}{2} = y/x$$

$$x\frac{(e^{\frac{-(\ln(x)-C)}{2}} - 1)}{2} = y$$